JOURNAL OF APPROXIMATION THEORY 13, 305–315 (1975)

Convergence of Positive Linear Operators on $C(X)$

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Communicated by P. L. Butzer

DEDICATED TO PROFESSOR G. G. LORENTZ ON THE OCCASION OF HIS SIXTY-FIFTH BIRTHDAY

INTRODUCTION

Bernstein polynomials play an important role in various areas of mathematics. They provide a useful tool in the analysis of numerous problems while also furnishing a source of ideas for further research. The book of G. G. Lorentz [7] provides an excellent source for many of their attractive properties. In this paper we are concerned with two basic properties of the Bernstein polynomials

$$B_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1 - x)^{n-k}, f \in C[0, 1], \qquad x \in [0, 1].$$

The first is the fact that $B_n(f, x)$ converges uniformly to f and the second is the fact that the rate of convergence cannot exceed $1/n$, except when f is a linear function. This latter statement expresses the fact that the Bernstein polynomials are saturated with order $1/n$. The proof, of these results are known to depend only on the behavior of B_n on the subspace of quadratic polynomials and the observation that B_n is a positive linear operator for all n [2, 6]. Thus the convergence of the Bernstein polynomials can be proved by applying Korovkin's theorem while their saturation is proved by the "parabolic method" of Bajšanski and Bojanic [2]. Here we will give a general setting for these two results. This setting will follow naturally from a study of the following question suggested in [8] (see also [9]).

PROBLEM 1. Given a compact set X and a set of "test functions" S (always assumed to be a closed subspace of $C(X)$ containing a positive function), describe all nonnegative operators T on $C(X)$ with the property that whenever $\{T_n\}$ is a sequence of nonnegative operators converging to T on S it follows that $\lim_{n\to\infty} T_n f = Tf$ for all $f \in C(X)$.

We note that in the case that the identity operator I has the property described above, the set S is called a Korovkin subspace of $C(X)$.

Here we will denote the dual of $C(X)$ (Radon measures on X) by $M(X)$ while $M^+(X)$ will denote the cone of positive Radon measures on X. In Section 1 we show that solution of the above problem requires the identification of all elements of $M^+(X)$ which are uniquely determined by S. We denote this set by $U(S)$. Thus $\mu \in U(S)$ provided that whenever $\nu \in M^+(X)$ with $\mu(f) = \nu(f)$ for all $f \in S$ then $\mu = \nu$. In Section 2 we give a characterization of $U(S)$ while Section 3 contains a closer discussion of $U(S)$ when S is a subspace of "parabolic functions" (see Section 3 for their definition). Section 3 also contains some applications to fixed points and saturation of positive linear operators. It is in Section 3 that we generalize and unify the two properties of the Bernstein polynomials mentioned earlier. We conclude the paper with some remarks about possible extensions of our results.

1. Convergence of Positive Linear Operators

Let S be a closed linear subspace of $C(X)$ which contains a positive function. We define $K(S)$ to be the class of all nonnegative linear operators T on $C(X)$ with the property that if $\{T_\lambda\}$ is a net of positive linear operators which converge to T on S then $\{T_\lambda\}$ converges to T on $C(X)$.

The use of nets rather than sequences as described in the introduction allows us to state Theorem 1.1 below without the hypothesis of metrizability on the compact set X. When X is first countable this distinction is unnecessary.

We will use the symbol ϵ_x to denote the Dirac measure defined by $\epsilon_x(f) = f(x)$, $f \in C(X)$.

THEOREM 1.1. *Let X be a compact Hausdorff space and S a closed linear subspace of $C(X)$ which contains a positive function. Then $T \in K(S)$ if and only if $\epsilon_x \circ T \in U(S)$ for all $x \in X$.*

Proof. Suppose there exists a point $y \in X$ such that $\epsilon_y \circ T \notin U(S)$. We will construct a net of positive linear operators converging to T on S but not on $C(X)$. From our hypothesis there exists a $\mu \in M^+(X)$ and a $g \in C(X)$ such that

$$T(g, y) \neq \mu(g) \qquad \text{and} \qquad T(f, y) = \mu(f), \qquad f \in S. \tag{1.1}$$

Let $\{V_\lambda \mid \lambda \in \Lambda\}$ be the set of all open neighborhoods containing y. For every $\lambda \in \Lambda$, let f_λ be a function in $C(X)$ which satisfies the following conditions

$$0 \leqslant f_\lambda(x) \leqslant 1, \qquad x \in X, \tag{1.2}$$

and

$$f_\lambda(x) = 0, \qquad x \notin V_\lambda \qquad \text{and} \qquad f_\lambda(y) = 1. \tag{1.3}$$

We define for $\lambda \in \Lambda$ the mapping

$$T_\lambda f = (1 - f_\lambda)\, Tf + f_\lambda \mu(f), \qquad f \in C(X). \tag{1.4}$$

Obviously, $\{T_\lambda \mid \lambda \in \Lambda\}$ is a net of positive linear operators on $C(X)$ which in view of (1.1)–(1.3) satisfies the inequality

$$| T_\lambda(f, x) - T(f, x)| \leqslant \sup_{x \in V_\lambda} | T(f, x) - T(f, y)| , \tag{1.5}$$

for all $f \in S$, $x \in X$. Thus $\{T_\lambda\}$ converges to T on S. But $T_\lambda(g, y) = \mu(g) \neq T(g, y)$ for $\lambda \in \Lambda$ which means $\{T_\lambda\}$ does not converge to T on $C(X)$. Thus we conclude $T \notin K(S)$.

Conversely, suppose $T \notin K(S)$. By virtue of the compactness of X, there exists a net $\{T_\lambda \mid \lambda \in \Lambda\}$ of positive linear operators which converges to T on S, a net $\{x_\lambda\}$ of points in X which converge to some $y \in X$, a function $h \in C(X)$ and an $\epsilon_0 > 0$ such that

$$| T_\lambda(h, x_\lambda) - T(h, x_\lambda)| \geqslant \epsilon_0 , \qquad \lambda \in \Lambda. \tag{1.6}$$

Since S contains a positive function the net $\{\epsilon_{x_\lambda} \circ T_\lambda \mid \lambda \in \Lambda\}$ is a norm bounded subset of $M^+(X)$. This implies that there exists a subnet $\{\epsilon_{x_{\lambda'}} \circ T_{\lambda'}\}$ which converges weakly to some $\nu \in M^+(X)$. ν and $\epsilon_y \circ T$ agree on S because by hypothesis $\{T_\lambda\}$ converges to T on S. But it follows from (1.6) that $\epsilon_y \circ T \neq \nu$ and so $\epsilon_y \circ T \notin U(S)$. This completes the proof.

Remark. Theorem 1 can be generalized in the following way. For any weakly closed convex subset L of $M(X)$ consider the set of operators on $C(X)$ such that $\epsilon_x \circ T \in L$ for all $x \in X$. Theorem 1 remains valid if we replace the class of positive linear operators by the above class induced by L and $M^+(X)$ by L, wherever they appear in Theorem 1.1, and in the definition of $K(S)$ and $U(S)$. In addition to the case $L = M^+(X)$, Lorentz considers in [8] the two choices $L_1 = \{\mu \mid \mu \in M(X), |\mu|(X) \leqslant 1\}$ and $L_1^+ = L_1 \cap M^+(X)$.

Perhaps the most interesting example of subspaces to consider are those which are finite dimensional. If we specialize Theorem 1.1 to a finite dimensional linear subspace, $\dim S = N$, we conclude that every $T \in K(S)$ is of the form

$$T(f, x) = \sum_{j=1}^{N} \lambda_j(x)\, f(a_j(x)), \tag{1.6}$$

where

$$\lambda_j(x) \geqslant 0, \qquad a_j(x) \in X, \qquad j = 1,..., N, \qquad x \in X.$$

This result follows from the fact that every $\mu \in M^+(X)$ can be represented on S as a nonnegative sum of at most N Dirac measures $\mu(f) = \sum_{j=1}^{N} \lambda_j \epsilon_{x_j}(f)$,

$f \in S$. This fact is sometimes referred to as Tchakaloff's theorem. Thus, for finite dimensional subspaces it is possible to obtain results analogous to Korovkin's theorem only for the operators in (1.6). When X is connected we may reduce the support of T in (1.6) to $N - 1$) (a consequence of Fenchel's theorem). This number can not in general be reduced further even when X is assumed to be convex. We showed in [10] that the operator defined by $T(f, x) = (1 - x)f(0) + xf(1)$, $x \in [0, 1]$, $f \in C[0, 1]$ is in $K(S)$ for $S =$ quadratic polynomials. This positive operator is supported on two points while the dimension of S is three. We will say more about this code in Section 3. Let us now present a characterization of $U(S)$.

2. A Characterization of $U(S)$

For any $\mu \in M^+(X)$ and $\omega \in C(X)$ we define

$$\underline{\omega}_S = \underline{\omega}_S(\mu) = \sup_{\substack{g \leq \omega \\ g \in S}} \mu(g) \tag{2.1}$$

and

$$\bar{\omega}_S = \bar{\omega}_S(\mu) = \inf_{\substack{g \geq \omega \\ g \in S}} \mu(g). \tag{2.2}$$

Since S contains a positive function it follows that $-\infty < \underline{\omega}_S \leq \bar{\omega}_S < +\infty$. Furthermore, the definition of (2.1) and (2.2) yield the inequalities

$$\underline{\omega}_S(\mu) \leq \mu(\omega) \leq \bar{\omega}_S(\mu). \tag{2.3}$$

We may interpret (2.3) as stating that the number $\nu(\omega)$ where ν is any extension of μ from S to $S_\omega = [\omega, S]$, the linear subspace spanned by ω and S, lies in the closed interval $[\underline{\omega}_S(\mu), \bar{\omega}_S(\mu)]$. Conversely, the linear functional defined

$$F_c(g + \alpha\omega) = \mu(g) + \alpha c, \qquad g \in S, \qquad \alpha \in R, \tag{2.3}$$

is a positive linear functional on S_ω if and only if $c \in [\underline{\omega}_S(\mu), \bar{\omega}_S(\mu)]$. Furthermore, since S contains a positive function, F_c can be extended to a positive measure ν_c on $C(X)$. These remarks are basic results from moment theory and can be found in [3] or [5]. They enable us to establish the following characterization of $U(S)$.

Theorem 2.1. *$\mu \in U(S)$ if and only if $\underline{\omega}_S(\mu) = \bar{\omega}_S(\mu)$ for all $\omega \in C(X)$.*

Proof. If $\mu \notin U(S)$ then there exists a $\nu \in M^+(X)$ and a $\omega \in C(X)$ such that $\nu(g) = \mu(g)$, $g \in S$ and $\nu(\omega) \neq \mu(\omega)$. Therefore, the interval $[\underline{\omega}_S(\mu), \bar{\omega}_S(\mu)]$

contains two distinct points which implies $\underline{\omega}_S(\mu) \neq \bar{\omega}_S(\mu)$. Conversely, suppose $\underline{\omega}_S(\mu) \neq \bar{\omega}_S(\mu)$ for some $\omega \in C(X)$. Choose two distinct numbers c and d in the interval $[\underline{\omega}_S(\mu), \bar{\omega}_S(\mu)]$. The measures ν_c and ν_d are distinct extensions of μ and so $\mu \notin U(S)$.

If we specialize Theorem 2.1 to $\mu = \epsilon_x$, $x \in X$, we obtain from Theorem 1.1 a characterization of a Korovkin set which is due to Berens and Lorentz [4].

COROLLARY 1.1. *S is a Korovkin subspace if and only if for every $\omega \in C(X)$ and $x \in X$*

$$\inf_{\substack{g \geqslant \omega \\ g \in S}} g(x) = \sup_{\substack{g \leqslant \omega \\ g \in S}} g(x).$$

3. $U(S)$ When S is a Subspace of Parabolic Functions

The simplest example of a Korovkin subspace for the space $C[0, 1]$ is the subspace of quadratic polynomials.

In [10] we show that $U(S)$ where S is the subspace of quadratic polynomials, consists precisely of nonnegative multiples of any Dirac measure ϵ_x, $x \in [0, 1]$ or "boundary measures" (measures supported on the extreme points of $[0, 1]$), $(1 - x)\,\epsilon_0 + x\epsilon_1$, $x \in [0, 1]$. In this section we give a generalization of this result. To do this we will rely on several results which can be found in [1].

E will denote a locally convex linear topological space and K a compact convex subset of E. $A(K)$ is the subspace of $C(K)$ consisting of all continuous affine function on K. Let $\partial_e K$ denote the set of extreme points of K. It is proved in [1] that

$$U(A(K)) = \{\lambda\epsilon_x \mid \lambda \in R^+, x \in \partial_e K\}. \tag{3.1}$$

As a consequence of (3.1) and Theorem 1 we see that any positive linear operator T which preserves affine functions satisfies the relation

$$\epsilon_x \circ T = \epsilon_x, \qquad x \in \partial_e K.$$

Another obvious conclusion of (3.1) is that $A(K)$ is not a Korovkin subspace of $C(K)$. However, we can remedy this by adding just one more function to $A(K)$. This leads use to define "parabolic functions" as any subspace of $C(K)$ of the form $S = S[\phi] = [\phi, A(K)]$ where ϕ is a strictly convex continuous function of K.

Recall that every convex function has a right Gateaux derivative, given by

$$D\phi(x; y) = \lim_{\lambda \to 0^+} \frac{\phi(x + \lambda y) - \phi(x)}{\lambda} = \inf_{\lambda > 0} \frac{\phi(x + \lambda y) - \phi(x)}{\lambda}$$

for all x, y such that $x \in K$, $x + y \in K$. We will say that ϕ is smooth provided that for all $x \in K$ the mapping $y \to D\phi(x; y - x)$ is in $A(K)$. Any smooth convex function has the property that it coincides with its lower envelope,

$$\phi(x) = \check{\phi}(x) = \max\{a(x) \mid a \leqslant \phi, a \in A(K)\}. \tag{3.2}$$

We may now prove the following result.

PROPOSITION 3.1. *If ϕ is a strictly convex smooth function then $\{\lambda\epsilon_x \mid \lambda \in R^+, x \in K\}$ is contained in $U(S[\phi])$.*

Proof. Given any $x \in K$ we conclude from (3.2) that there exists an affine function $a \in A(K)$ such that $a \leqslant \phi$ and $a(x) = \phi(x)$. The strict convexity of ϕ implies that $a(y) < \phi(y)$ for all $y \neq x$. Now, let μ be any measure in $M^+(X)$ which agrees with $\lambda\epsilon_x$ on the subspace $S[\phi]$. Then $\mu(\phi - a) = 0$. This implies that $\mu = \mu(1)\,\epsilon_x$. To arrive at this conclusion we use the following property of positive measures. If $\mu(h) = 0$ for some $h \in C(K)$ which is nonnegative on K then $\mu(f) = 0$ for all f such that $\{x \mid h(x) = 0\} \subseteq \{x \mid f(x) = 0\}$. Since $1 \in S[\phi]$ we also have $\lambda = \mu(1)$. Thus $\lambda\epsilon_x \in U(S[\phi])$.

Before we can identify other members of $U(S[\phi])$ we first describe several results from [1].

A boundary measure is defined to be any $\mu \in M^+(X)$ which is "supported" on the extreme points of K. It can be shown that

$$\partial_e K = \bigcap_{f \in C(K)} \{x \mid f(x) = \hat{f}(x)\} \tag{3.3}$$

where $\hat{f}$ is the upper envelope of f defined by

$$\hat{f}(x) = \inf\{a(x) \mid a \geqslant f, a \in A(K)\}. \tag{3.4}$$

Therefore μ is defined to be a boundary measure if and only if

$$\mu(f - \hat{f}) = 0, \qquad \text{for all } f \in C(K). \tag{3.5}$$

Boundary measures can also be characterized in terms of the following ordering induced by the set $P(K)$ of all continuous convex functions on K.

$$\mu < \nu \Leftrightarrow \mu(f) \leqslant \nu(f), \qquad f \in P(K). \tag{3.6}$$

It can be shown that μ is a boundary measure if and only if it is a maximal element with respect to this ordering. A positive measure μ is said to represent $x \in K$ if $\epsilon_x(a) = \mu(a)$, for all $a \in A(K)$. Every positive measure μ represents exactly one point in K. This point is called the resultant of μ. A compact convex set K is called a Bauer simplex if and only if $\partial_e K$ is closed and every $x \in K$ is represented by a unique positive boundary measure Π_x.

PROPOSITION 3.2. *Suppose K is a Bauer simplex and ϕ is a strictly convex continuous function on K, then $\{\lambda\Pi_x \mid \lambda \in R^+, x \in K\} \subseteq U(S[\phi])$.*

Proof. Suppose $x \in K$, $\mu \in M^+(X)$ and $\mu(g) = \lambda\Pi_x(g)$, $g \in S[\phi]$. Since K is a Bauer simplex, we have $\hat{\phi} \in A(K)([1])$. Since Π_x is a boundary measure we have from (3.5)

$$\mu(\phi - \hat{\phi}) = \lambda\Pi_x(\phi - \hat{\phi}) = 0.$$

However, the strict convexity of ϕ implies

$$\{x \mid \phi(x) = \hat{\phi}(x)\} = \partial_e K \subseteq \{x \mid f(x) = \hat{f}(x)\}, \qquad f \in C(K)([1]).$$

Thus $\mu(f - \hat{f}) = 0$ for all $f \in C(K)$ and so μ must be a boundary measure. Hence, $\mu = \lambda\Pi_y$ where y is the resultant of μ. This follows from the assumption that K is a Bauer simplex. Since $A(K)$ separates points, we obtain $x = y$. This completes the proof.

Our next proposition shows that every element in $U(S)$ is a convex combination of a Dirac measure and a boundary measure.

PROPOSITION 3.3. *Suppose ϕ is a continuous convex function on a compact convex set. Then every $\mu \in U(S[\phi])$ with $\mu(1) = 1$ has the form $\mu = (1 - \lambda)\,\epsilon_x + \lambda\Pi_x$, for some $\lambda \in [0, 1]$ where x is the resultant μ. Furthermore, if $0 < \lambda < 1$ then $x \in \overline{\partial_e K} - \partial_e K$.*

Proof. We have from (3.6) the inequalities

$$\epsilon_x(\phi) \leqslant \mu(\phi) \leqslant \Pi_x(\phi) \tag{3.7}$$

where Π_x is a boundary measure representing x, the resultant of μ. Thus there exists a $\lambda \in [0, 1]$ such that $\mu(\phi) = (1 - \lambda)\,\epsilon_x(\phi) + \lambda\Pi_x(\phi)$. It follows that μ and $(1 - \lambda)\,\epsilon_x + \lambda\Pi_x$ agree on the subspace $U(S[\phi])$. Thus we conclude that $\mu = (1 - \lambda)\,\epsilon_x + \lambda\Pi_x$.

Suppose that $0 < \lambda < 1$ then clearly

$$\epsilon_x(\phi) < \mu(\phi) < \Pi_x(\phi). \tag{3.8}$$

Therefore $x \notin \partial_e K$ since otherwise from (3.1) we would obtain $\epsilon_x = \mu$. Hence there exists a nonzero $y \in E$ such that $x + \epsilon y \in K$ for all ϵ sufficiently small. By virtue of (3.8) we obtain

$$\tfrac{1}{2}(\epsilon_{x+\epsilon y}(\phi) + \epsilon_{x-\epsilon y}(\phi)) < \mu(\phi) < \Pi_x(\phi). \tag{3.9}$$

for some positive ϵ. Now we may argue just as before to conclude that there exists an $\alpha \in (0, 1)$ such that $\mu = (1 - \alpha)[\frac{1}{2}(\epsilon_{x+\epsilon y} + \epsilon_{x-\epsilon y})] + \alpha\Pi_x$. However, we have already proved $\mu = (1 - \lambda)\, \epsilon_x + \lambda \Pi_x$. Thus it follows that $x \in \overline{\partial_e K}$. This completes the proof.

We remark that there is an example (kindly communicated to us by A. Gleit) which shows that in general no stronger conclusion can be made about an element in $U(S[\phi])$. Let V denote the subspace of $C[0, 1]$ defined by $V = \{f \mid f \in C[0, 1], f(\frac{1}{2}) = \frac{1}{2}(f(0) + f(1))\}$. For our compact convex set K we choose the state space of V, namely $K = \{\mu \mid \mu \in V^*, \mu \geqslant 0, \mu(1) = \|\mu\|\}$. It can easily be verified that $\{\epsilon_{1/2}\} = \overline{\partial_e K} - \partial_e K$; this leads us to the desired example.

Combining Propositions 3.1–3.3 we obtain a complete description of $U(S[\phi])$ when K is a Bauer simplex.

THEOREM 3.1. *Suppose ϕ is a strictly convex smooth function on a Bauer simplex K. Then*

$$U(S[\phi]) = \{\lambda \epsilon_x \mid \lambda \in R^+, x \in K\} \cup \{\lambda \Pi_x \mid \lambda \in R^+, x \in K\}.$$

Let us specialize Theorem 3.1 to a finite dimensional simplex. In this case, $E = R^N$ and $K = \Delta_N = \{x \mid x \in R^N, \sum_{i=1}^N x_i \leqslant 1, x_i \geqslant 0, i = 1,..., N\}$. The extreme points $\partial_e K = \{e_0, e_1, ..., e_N\}$ are defined by $(e_i)_j = \delta_{ij}$, $i = 0, 1,..., N$, $j = 1,..., N$. If we choose $\phi(x) = x \cdot x = \sum_{i=1}^N x_i^2$ then

$$U(S[\phi]) = \{\lambda \epsilon_x \mid \lambda \in R^+, x \in K\} \cup \left\{\lambda\left(1 - \sum_{i=1}^N x_i\right) \delta_{e_0} + \lambda\sum_{i=1}^N x_i \delta_{e_i} \;\lambda \in R^+, x \in K\right\}.$$

The case $N = 1$ was previously referred to in the beginning of this section.

We give two applications of the results presented in this section. The first concerns fixed points of nonnegative operators. In the following discussion we always assume K is a Bauer simplex.

Let T be a nonnegative linear operator on $C(K)$ which preserves affine functions $T(a) = a$, $a \in A(K)$. Define $S(K) = \{\phi \mid \phi \geqslant 0, \phi$ strictly concave, smooth and continuous on $K\}$.

Set

$$\lambda(T, \phi) = \sup_{\substack{x \in K \\ \phi(x) \neq 0}} T(\phi, x)/\phi(x)$$

and

$$\lambda(T) = \inf_{\phi \in S(K)} \lambda(T, \phi).$$

Note that $0 \leqslant \lambda(T) \leqslant 1$.

THEOREM 3.2. *Suppose T is a nonnegative operator on $C(K)$ which preserves affine functions. If $\lambda(T) < 1$ then the only fixed points of T are in $A(K)$.*

Proof. Since $\lambda(T) < 1$ there exists a $\phi \in S(K)$ such that for $\bar{\lambda} = \frac{1}{2}(1 + \lambda(T))$ (<1) we have

$$T(\phi, x) \leqslant \bar{\lambda}\phi(x), \qquad \text{for} \quad x \in K.$$

Thus $\lim_{k\to\infty} T^k(\phi, x) = 0$ uniformly in $x \in K$. However, since $\bar{\lambda} < 1$ we necessarily have $\phi(x) = 0$ for all $x \in \partial_e K$. This implies that $\lim_{k\to\infty} T^k g = \tilde{g}$, for all $g \in S[\phi]$, where $\tilde{g}$ is defined by $\Pi_x(g) = \tilde{g}(x)$, $x \in K$. The correspondence $g \to \tilde{g}$ is a positive linear operator on $C(K)$ whose range is contained in $A(K)$([1]). Thus by Theorem 1.1 and Proposition 3.2 we have for all $f \in C(X)$

$$\lim_{k\to\infty} T^k f = \tilde{f}.$$

Hence, if h is a fixed point of T in $C(K)$ we obtain $h = \tilde{h} \in A(K)$.

Using the idea employed in Theorem 3.2 we may prove a general "little o" saturation theorem for positive operators on a Bauer simplex.

THEOREM 3.3. *Let $\{T_n\}$ be a sequence of positive linear operators on $C(K)$ which preserves affine functions and satisfy the condition*

$$\lambda(T_n, \phi) < 1, \qquad \limsup_{n\to\infty} \lambda(T_n, \phi) = 1 \tag{3.10}$$

for some $\phi \in S(K)$. Then $\{T_n\}$ is saturated with order $1 - \lambda(T_n, \phi)$. Thus $T_n f - f = o(1 - \lambda(T_n, \phi))$ implies $f = \tilde{f} \in A(K)$.

Proof. Our hypothesis implies that the norm of T_n is one. Therefore the identity

$$T_n^k f - f = \sum_{j=0}^{k-1} T_n^j (T_n f - f)$$

implies that

$$\| T_n^k f - f \| \leqslant k \| T_n f - f \|. \tag{3.11}$$

This inequality and our hypothesis imply that there exists a sequence of integers $\{k_n\}$, such that $\lim_{n\to\infty} k_n(1 - \lambda(T_n, \phi)) = \infty$ and $\lim_{n\to\infty} T_n^{k_n} f = f$. However, from (3.10) we conclude that there exists a $\phi \in S(K)$ and a subsequence $\{n'\}$ such that $\lim_{n\to\infty} \lambda(T_{n'}, \phi) = 1$ and $T_{n'}\phi \leqslant \lambda(T_{n'}, \phi)\, \phi$. Thus it follows that $\lim_{n\to\infty} T_{n'}^{k_{n'}} \phi = 0$. Just as before we conclude that $\lim_{n\to\infty} T_{n'}^{k_{n'}} f = \tilde{f}$. Thus $f = \tilde{f}$ and the theorem is proved.

EXAMPLE 3.2. As an application of Theorem 3.3 let us look at the Bernstein polynomials defined on the N-simplex Δ_N,

$$B_n(f, x) = \sum_{\frac{\nu}{n} \in \Delta_N} f\left(\frac{\nu}{n}\right) \binom{\nu}{n} \phi_\nu(x),$$

where

$$\binom{\nu}{n} = n!/\nu_1! \cdots \nu_N!\,(n - \nu_1 \cdots - \nu_N)!$$

and

$$\phi_\nu(x) = x_1^{\nu_1} \cdots x_N^{\nu_N}(1 - x_1 - \cdots - x_N)^{n-\nu_1-\cdots-\nu_N}.$$

B_n is a positive linear operator $C(\Delta_N)$ which preserves affine functions. Furthermore, $B_n\phi = (1 - 1/n)\,\phi$ where $\phi(x) = \sum_{i=1}^N x_i(1 - x_i)$. Thus from Theorem 3.3, $\{B_n\}$ is saturated with order $1/n$.

It is interesting to note that for $N > 1$ there is no local saturation result for the Bernstein polynomials on the simplex Δ_N, while for $N = 1$ such a result is known to be true. For details on this matter see [2] and [9].

We end this paper with a question which is concerned with a possible extension of Theorem 3.1.

Let C be a cone contained in $C(X)$. C induces an ordering in $M^+(X)$ defined by

$$\mu < \nu \text{ if and only if } \mu(f) \leqslant \nu(f), \qquad f \in C.$$

Suppose this ordering admits maximal and minimal elements. Thus for every μ there exists a minimal $\underline{\mu}$ and maximal $\bar{\mu}$ such that $\underline{\mu} < \mu < \bar{\mu}$. Choose some $\phi \in C$ and let $S[\phi]$ be the subspace spanned by ϕ and the base of the cone $C \cap (-C)$. When is it true that

$$U(S[\phi]) = \{\underline{\mu} \mid \mu \in M^+(X)\} \cup \{\bar{\mu} \mid \mu \in M^+(X)\}\,? \tag{3.12}$$

Our paper [10] as well as Theorem 3.1 give instances in which (3.12) is valid. It would be interesting to have other examples of (3.12).

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